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Superalgebra and the spherical model of a spin glass†

Alba Theumann

Instituto de Física, Universidade Federal do Rio Grande do Sul, 90049 Porto Alegre, RS, Brazil

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Abstract. Supermathematics methods are used in the exact solution of the mean spherical model of a spin glass, first solved by Kosterlitz, Thouless and Jones. Our analysis shows that the model is pathological, in the sense that the exact solution reduces to that obtained from two decoupled replicas. We show explicitly that even one replica gives the exact result, which means that the 'wrong' solution of taking the configurational average of the partition function is correct in this model.

1. Introduction

The spherical model of a spin glass has been introduced and solved exactly by Kosterlitz *et al* (1976) using Wigner's exact result for the density of eigenvalues of a large random matrix (Mehta 1967). They also found that the replica method with the replica symmetric SK approximation (Sherrington and Kirkpatrick 1975) reproduced the exact results.

In a later publication, de Almeida *et al* (1978) have shown that the SK solution for the m-vector model spin glass is unstable below T_c for any finite m but it is the only stable solution when $m \to \infty$ in the spherical model limit, in agreement with the results of Kosterlitz *et al.*

In the present paper we study the mean spherical model (Joyce 1972) of a spin glass by using supermathematics (Efetov 1983) with the purpose of having a better understanding of a new technique by testing it on a problem that has a well known exact solution. We explicitly show in this paper that the *exact* solution reduces to *two decoupled* 'replicas', thus explaining why the SK method is exact in this model: any number n of replicas will decouple, reproducing n times the exact free energy, the limit $n \to 0$ being trivial in this case. The model is thus pathological in the sense that the results of Kosterlitz *et al* are also obtained for $n = 1$ and this means that the *wrong* solution for the thermodynamic potential $\beta\Omega = -\ln\langle Q\rangle_{CA}$, where Q is the partition function and $\langle\ \rangle_{CA}$ indicates a configurational average, becomes *exact* in the random spherical model with the SK type of interactions, as is shown in § 2. In a recent investigation on the random spherical model, Pastur (1982) discussed the problematic nature of the long-range interaction case.

Although the Grassmann variables technique may look an exceedingly complicated method when applied to the present problem, one must take into account that we are

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in fact giving an alternative derivation of Wigner's semicircular law, as it is shown at the end of § 2. The elegance and simplicity of the solution obtained by Kosterlitz *et al* (1976) stems from the fact that they used Wigner's law from the start. The semicircular law has also been derived by Edwards and Jones (1976) and lately by Verbaarschot and Zirnbauer (1984) using the replica method.

Theoretical physicists have long recognised the importance of Grassmann anticommuting c variables in the theory of condensed matter because they give a natural representation of fermion fields in a functional integral formalism (Bell 1962, Berezin 1966, Edwards and Sherrington 1967). More recently the Grassmann variables technique has also been applied in a wider context. Supersymmetry methods have proved to be a powerful tool in the problem of spins in a random field (Parisi and Sourlas 1979), to calculate the exact density of states of electrons in a random potential and a strong magnetic field (Wegner 1983, Brezin *et al* 1984, Klein and Perez 1984) and in the study of random two-dimensional Ising models (Jug 1984). Verbaarschot and Zirnbauer (1985) have shown the advantage of using the method of superfields in a problem where the replica method fails to give the correct answer. A recent review in supermathematics and its applications to the theory of disordered metals is given by Efetov (1983). What makes supermathematics a particularly useful method to study random systems is that it allows us to write a generating functional $Z(\{h\})$, for a set of auxiliary fields $\{h\}$, such that $Z(0)$ equals unity while the correlation functions are obtained from functional differentiation with respect to $\{h\}$. It then follows that the configurational average of $Z(\{h\})$ can be directly calculated without using the replica method (De Dominicis 1978). This approach is followed in § 2 where the generating functional $Z(\{h\})$ is written as an integral over two commuting fields $\sigma_{i\alpha}$, $\alpha = 1$ or 2, and a complex Grassmann field χ_i, at each lattice site i. Explicit calculations give the results mentioned at the beginning of this section.

2. General formalism and exact solution

The model we consider has continuous spins σ_j at lattice sites j coupled by random exchange constants J_{ij} between all distinct pairs of spins with a Gaussian probability distribution of zero mean and variance J^2/N, for the total number of lattice sites, N.

The partition function in the mean spherical model (Joyce 1972) is given at zero field by

$$Q(\mu) = \int_{-\infty}^{\infty} \prod_i \frac{\mathrm{d}\sigma_i}{\sqrt{\pi}} \exp\left(\tfrac{1}{2}\sum_{i,j} K_{ij}\sigma_i\sigma_j - \mu \sum_i \sigma_i^2\right)$$

$$= \frac{1}{(|\mathbf{\Gamma}|)^{1/2}} \tag{1}$$

for $\mathbf{\Gamma}$ a $N \times N$ random matrix with elements

$$\Gamma_{ij} = \mu\delta_{ij} - \tfrac{1}{2}K_{ij} \tag{2}$$

$$K_{ij} = \beta J_{ij} \qquad K_{ii} = 0 \tag{3}$$

and $\beta = (k_\mathrm{B}T)^{-1}$. In the uniform model the 'chemical potential' μ ensures that, on the average, the σ_j lie on a sphere of radius N. In the present random case we impose the spherical condition on the configurational average:

$$-\frac{\partial}{\partial\mu}\langle \ln Q(\mu)\rangle_{\mathrm{CA}} = \sum_i \langle\langle\sigma_i^2\rangle_{\mathrm{TA}}\rangle_{\mathrm{CA}} = N \tag{4}$$

with the obvious notation

$$\langle\ldots\rangle_{\mathrm{TA}} = \frac{1}{Q(\mu)}\int\prod_i \frac{\mathrm{d}\sigma_i}{\sqrt{\pi}}\exp\left(-\sum_{i,j}\Gamma_{ij}\sigma_i\sigma_j\right)(\ldots) \tag{5}$$

and $\langle\ldots\rangle_{\mathrm{CA}}$ indicates a configurational average over the random J_{ij}. The problem then reduces to the calculation of $\langle \ln Q(\mu)\rangle_{\mathrm{CA}}$ and this was achieved by Kosterlitz *et al* (1976) using Wigner's semicircular law.

Here we use an alternative method. We introduce a system of N complex anticommuting Grassmann variables χ_i, one at each lattice site, with the properties (Efetov 1983)

$$\{\chi_i, \chi_j\} = \{\chi_i^*, \chi_j^*\} = \{\chi_i, \chi_j^*\} = 0 \qquad \chi_i^2 = 0 \tag{6}$$

$$\int \mathrm{d}\chi_i\,\chi_j = \int \mathrm{d}\chi_i^*\chi_j^* = \delta_{ij} \qquad \int \mathrm{d}\chi_i = \int \mathrm{d}\chi_i^* = 0 \tag{7}$$

$$\int \prod_i \mathrm{d}\chi_i^*\,\mathrm{d}\chi_i \exp\left(-\sum \Gamma_{ij}\chi_i^*\chi_j\right) = |\boldsymbol{\Gamma}|. \tag{8}$$

In the following we indicate Grassmann variables with Greek letters. Next we consider the functional:

$$Z(\{h\}) = \left\langle \frac{Q(\mu;\{h_1\})Q(\mu;\{h_2\})}{Q^2(\mu)}\right\rangle_{\mathrm{CA}} \tag{9}$$

with $Q(\mu)$ as in (1) and

$$Q(\mu,\{h_\alpha\}) = \int_{-\infty}^{\infty}\prod_i \frac{\mathrm{d}\sigma_i}{\sqrt{\pi}}\exp\left(-\sum_{i,j}\Gamma_{ij}\sigma_i\sigma_j + \sum_i h_{i\alpha}\sigma_i\right) \tag{10}$$

for $h_{i\alpha}$ a set of two auxiliary fields at sites i, $\alpha = 1$ or 2. $Z(\{h\})$ has the following properties:

$$Z(0) \equiv 1 \tag{11}$$

$$\left(\frac{\partial^2 Z}{\partial h_{i\alpha}^2}\right)_{\{h\}=0} = \left\langle\left(\frac{1}{Q}\frac{\partial^2 Q(\mu,\{h_\alpha\})}{\partial h_{i\alpha}^2}\right)_{\{h\}=0}\right\rangle_{\mathrm{CA}} = \langle\langle\sigma_i^2\rangle_{\mathrm{TA}}\rangle_{\mathrm{CA}} \tag{12}$$

$$\left(\frac{\partial^2 Z}{\partial h_{i1}\,\partial h_{i2}}\right)_{\{h\}=0}\left\langle\left(\frac{1}{Q}\frac{\partial Q(\mu,\{h\})}{\partial h_i}\right)^2_{\{h\}=0}\right\rangle = \langle\langle\sigma_i\rangle_{\mathrm{TA}}^2\rangle_{\mathrm{CA}} \tag{13}$$

from the definition in equation (5). The thermodynamic potential per site is the solution of the differential equation, from equation (1):

$$\frac{\partial}{\partial\mu}(\beta\Omega(\mu)) = \frac{1}{N}\sum_i \langle\langle\sigma_i^2\rangle_{\mathrm{TA}}\rangle_{\mathrm{CA}} \tag{14}$$

with the boundary condition:

$$\lim_{\beta\to 0}[\beta\Omega(\mu)] = \tfrac{1}{2}\ln[\mu]. \tag{15}$$

Then all physical properties of the system can be obtained from a knowledge of $Z[\{h\}]$. In particular the spherical condition in (4) can be written from (12):

$$\frac{1}{N}\sum_i\left(\frac{\partial^2 Z(\{h\})}{\partial h_{i\alpha}^2}\right)_{\{h\}=0}=1 \qquad \alpha=1 \text{ or } 2 \tag{16}$$

the spin glass order parameter can be defined from (13) (Sherrington and Kirkpatrick 1975):

$$q_{\mathrm{SG}}=\frac{1}{N}\sum_i\left(\frac{\partial^2 Z(\{h\})}{\partial h_{i1}\,\partial h_{i2}}\right)_{\{h=0\}} \tag{17}$$

while the uniform susceptibility χ satisfies

$$\chi=\beta\frac{1}{N}\sum_{i,j}\left(\frac{\partial^2 Z(\{h\})}{\partial h_{i1}\,\partial h_{j1}}-\frac{\partial^2 Z(\{h\})}{\partial h_{i1}\,\partial h_{j2}}\right)_{\{h=0\}}$$

$$=\beta\frac{1}{N}\sum_{i,j}\langle\langle\sigma_i\sigma_j\rangle_{\mathrm{TA}}-\langle\sigma_i\rangle_{\mathrm{TA}}\langle\sigma_j\rangle_{\mathrm{TA}}\rangle_{\mathrm{CA}}. \tag{18}$$

From (1), (8) and (9) we can write

$$Z(\{h\})=\langle|\Gamma|Q(\mu,\{h_1\})Q(\mu,\{h_2\})\rangle_{\mathrm{CA}}$$

$$Z(\{h\})=\int_{-\infty}^{\infty}\sum_{i,\alpha}\frac{\mathrm{d}\sigma_{i\alpha}}{\pi^N}\int\prod_i \mathrm{d}\chi_i^*\,\mathrm{d}\chi_i\left\langle\exp\left[-\sum_{i,j}\Gamma_{ij}\left(\chi_i^*\chi_j+\sum_{\alpha=1}^{2}\sigma_{i\alpha}\sigma_{j\alpha}\right)+\sum_{i,\alpha}h_{i\alpha}\sigma_{i\alpha}\right]\right\rangle_{\mathrm{CA}}. \tag{19}$$

The Gaussian average over the random J_{ij} is now trivial to perform so we obtain

$$Z(\{h\})=\int_{-\infty}^{\infty}\prod_{i,\alpha}\frac{\mathrm{d}\sigma_{i\alpha}}{\pi^N}\int\prod_i \mathrm{d}\chi_i^*\,\mathrm{d}\chi_i\exp\left\{-\mu\sum_i\left(\chi_i^*\chi_i+\sum_{\alpha=1}^{2}\sigma_{i\alpha}\sigma_{i\alpha}\right)+\sum_{i,\alpha}h_{i\alpha}\sigma_{i\alpha}\right.$$

$$+\frac{K^2}{2N}\left[\frac{1}{2}\sum_\alpha\left(\sum_i\sigma_{i\alpha}^2\right)^2+\left(\sum_i\sigma_{i1}\sigma_{i2}\right)^2-\frac{1}{4}\left(\sum_i\chi_i^*\chi_i\right)^2\right.$$

$$\left.\left.+\sum_\alpha\left(\sum_i\chi_i^*\sigma_{i\alpha}\sum_j\chi_j\sigma_{j\alpha}\right)\right]-\frac{K^2}{4N}\sum_i(\sigma_{i1}^2+\sigma_{i2}^2+\chi_i^*\chi_i)^2\right\} \tag{20}$$

where

$$K=\beta J. \tag{21}$$

$Z(\{h\})$ is just the partition functional of two 'replicas' coupled by a Grassmann field. We notice first that the linear combination

$$\eta_\alpha=\sum\chi_i\sigma_{i\alpha} \tag{22}$$

is a Grassmann variable as

$$\eta_\alpha^2=\sum_{i\neq j}\chi_i\chi_j\sigma_{i\alpha}\sigma_{j\alpha}=0 \tag{23}$$

because the antisymmetry of the product $\chi_i\chi_j=-\chi_j\chi_i$, while $\Sigma_i\chi_i^*\chi_i$ is an ordinary c number:

$$\left(\sum_i\chi_i^*\chi_i\right)^2=\sum_{i\neq j}\chi_i^*\chi_i\chi_j^*\chi_j\neq 0. \tag{24}$$

We then use the Gaussian identity for an ordinary variable:

$$e^{a^2/N}=\int_{-\infty}^{\infty}\left(\frac{N}{\pi}\right)^{1/2}\mathrm{d}x\exp(-Nx^2+2xa) \tag{25}$$

and its Grassmann variable counterpart (Efetov 1983)

$$\exp\left(\frac{\eta^*\eta}{N}\right)=\int\frac{\mathrm{d}\theta^*\,\mathrm{d}\theta}{N}\exp[-(N\theta^*\theta+\theta^*\eta+\eta^*\theta)] \tag{26}$$

to write (20) in terms of a set of order parameter fields:

$$Z(\{h\})=\int_{-\infty}^{\infty}\frac{\mathrm{d}x_1\,\mathrm{d}x_2}{2\pi/N}\frac{\mathrm{d}q\,\mathrm{d}s}{\pi/N}\int\frac{\mathrm{d}\theta_1^*\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2^*\,\mathrm{d}\theta_2}{N^2}\times\exp\left[-N\left(\frac{x_1^2+x_2^2}{2}+q^2+s^2+\theta_1^*\theta_1+\theta_2^*\theta_2-\Lambda\right)\right] \tag{27}$$

where Λ is given by

$$e^{N\Lambda}=\int_{-\infty}^{\infty}\prod_{i,\alpha}\frac{\mathrm{d}\sigma_{i\alpha}}{\pi^N}\int\prod_i\mathrm{d}\chi_i^*\,\mathrm{d}\chi_i\exp\left\{-\sum_j\left[\mu\left(\chi_j^*\chi_j+\sum_{\alpha=1}^{2}\sigma_{j\alpha}\sigma_{j\alpha}\right)+\sum_\alpha h_{j\alpha}\sigma_{j\alpha}+\frac{K}{\sqrt{2}}\left(\sum_\alpha x_\alpha\sigma_{j\alpha}^2+2q\sigma_{j1}\sigma_{j2}+\mathrm{i}s\chi_j^*\chi_j+\sum_\alpha(\theta_\alpha^*\chi_j\sigma_{j\alpha}+\chi_j^*\sigma_{j\alpha}\theta_\alpha)\right)\right]\right\} \tag{28}$$

and we have neglected the last term in the exponent of (20) because it gives a contribution $O(1/N)$. Equation (28) splits into the product of N one-site quadratic integrals that are easily performed by using (1) and (8). The result can be expressed in terms of the 4×4 supermatrix:

$$\mathbf{A}=\begin{bmatrix}\mathbf{M} & -\boldsymbol{\theta}^+\\ -\boldsymbol{\theta} & \mathbf{S}\end{bmatrix} \tag{29}$$

with

$$\mathbf{M}=\begin{bmatrix}A_1 & -q\\ -q & A_2\end{bmatrix}\qquad \mathbf{S}=\begin{bmatrix}A_s & 0\\ 0 & A_s\end{bmatrix} \tag{30}$$

$$\boldsymbol{\theta}=\frac{1}{\sqrt{2}}\begin{bmatrix}\theta_1 & \theta_2\\ \theta_1 & \theta_2\end{bmatrix}\qquad \boldsymbol{\theta}^+=\frac{1}{\sqrt{2}}\begin{bmatrix}\theta_1^* & \theta_1^*\\ \theta_2^* & \theta_2^*\end{bmatrix}$$

$$A_s=z-\mathrm{i}s\qquad A_\alpha=z-x_\alpha\qquad \alpha=1,2$$
$$\mu=(K/\sqrt{2})z. \tag{31}$$

It is obtained that

$$\Lambda=\frac{\sqrt{2}}{4KN}\sum_j\bar{h}_j^+\mathbf{A}^{-1}\bar{h}_j+\tfrac{1}{2}\ln\{|\mathbf{S}||\mathbf{M}-\boldsymbol{\theta}^+\mathbf{S}^{-1}\boldsymbol{\theta}|^{-1}\} \tag{32}$$

where

$$\bar{h}_j=\begin{bmatrix}h_{j1}\\ h_{j2}\\ 0\\ 0\end{bmatrix}. \tag{33}$$

By introducing the 4×4 supermatrix:

$$\mathbf{X}=\begin{bmatrix} x_1 & q & \boldsymbol{\theta}^+ & \\ q & x_2 & & \\ & & \mathrm{is} & 0 \\ & \boldsymbol{\theta} & 0 & \mathrm{is} \end{bmatrix} \tag{34}$$

one can write (27) in supersymmetric form:

$$Z(\{h\})=\int \mathrm{D}\mathbf{X}\exp\{-\tfrac{1}{2}N\,\mathrm{Sup\,Tr}[\mathbf{X}^2+\ln(z-\mathbf{X})]\} \tag{35}$$

where D**X** is a shorthand notation for the element of volume in (27). The supertrace and superdeterminant of a supermatrix like **A** in (29) are defined by

$$\begin{aligned} &\mathrm{Sup\,Tr}\,\mathbf{A}=\mathrm{Tr}\,\mathbf{M}-\mathrm{Tr}\,\mathbf{S} \\ &\mathrm{Sup\,Det}\,\mathbf{A}=|S|\,|\mathbf{M}-\boldsymbol{\theta}^+\mathbf{S}^{-1}\boldsymbol{\theta}|^{-1} \\ &\mathrm{Sup\,Tr}\ln\mathbf{A}=\ln\mathrm{Sup\,Det}\,\mathbf{A}. \end{aligned} \tag{36}$$

We refer the reader to the article by Efetov (1983) for a detailed account on superalgebra. The method followed in the thermodynamic limit is by steepest descent in the ordinary and Grassmann variables. This would lead from (35) to the saddle-point matrix equation

$$\mathbf{X}=\tfrac{1}{2}(z-\mathbf{X})^{-1}=\tfrac{1}{2}\mathbf{A}^{-1}. \tag{37}$$

Although equation (37) is in a compact form, to invert a 4×4 supermatrix leads to complicated expressions. We prefer to take advantage of the fact that any function of a Grassmann variable is necessarily a first degree polynomial in that variable and to expand Λ in (32) as

$$\Lambda=[\Lambda]_{\{\theta\}=0}-\frac{1}{2A_\mathrm{s}}\mathrm{Tr}\,\boldsymbol{\theta}\mathbf{M}^{-1}\boldsymbol{\theta}^+. \tag{38}$$

Only terms that give a significant contribution in the thermodynamic limit have been kept in (38).

Terms in Λ proportional to $\theta_1^*\theta_1\theta_2^*\theta_2$ will give a contribution $\mathrm{O}(1/N)$ as can easily be checked from (27) and the integration rules in (7). Now $\bar{h}_j$ indicates a two-component column vector. The result for $Z(\{h\})$ is

$$Z(\{h\})=\int_{-\infty}^{\infty}\frac{\mathrm{d}x_1\,\mathrm{d}x_2}{2\pi/N}\frac{\mathrm{d}q\,\mathrm{d}s}{\pi/N}\exp\left(-NG+\frac{\sqrt{2}}{4K}\sum_j\bar{h}_j^+\mathbf{M}^{-1}\bar{h}_j\right)\left|1-\frac{1}{2A_\mathrm{s}}\mathbf{M}^{-1}\right| \tag{39}$$

where

$$G=\tfrac{1}{2}(x_1^2+x_2^2)+q^2+s^2-\ln A_\mathrm{s}+\tfrac{1}{2}\ln|\mathbf{M}|. \tag{40}$$

From (16) and (39) we obtain the leading term in the spherical condition:

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}q\,\mathrm{d}s}{2\pi^2/N^2}\,\mathrm{e}^{-NG}\left|1-\frac{1}{2A_\mathrm{s}}\mathbf{M}^{-1}\right|\frac{A_\alpha}{2|\mathbf{M}|}=\frac{K}{\sqrt{2}} \tag{41}$$

and from (17) the spin glass order parameter is given by

$$q_{\mathrm{SG}}=\frac{\sqrt{2}}{K}\int_{-\infty}^{\infty}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}q\,\mathrm{d}s}{2\pi^2/N^2}\,\mathrm{e}^{-NG}\left|1-\frac{1}{2A_\mathrm{s}}\mathbf{M}^{-1}\right|\frac{q}{2|\mathbf{M}|} \tag{42}$$

while

$$Z(0)=\int_{-\infty}^{\infty}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}q\,\mathrm{d}s}{2\pi^2/N^2}\,\mathrm{e}^{-NG}\left|1-\frac{1}{2A_{\mathrm{s}}}\mathbf{M}^{-1}\right|. \tag{43}$$

The integrals in (41)-(43) are performed by steepest descent with the result in the thermodynamic limit:

$$Z_{\mathrm{SP}}(0)=\mathrm{e}^{-NG_{\mathrm{SP}}}\left[\left(1-\frac{\bar{x}_1}{z-\mathrm{i}\bar{s}}\right)\left(1-\frac{\bar{x}_2}{z-\mathrm{i}\bar{s}}\right)-\frac{\bar{q}^2}{(z-\mathrm{i}s)^2}\right]\frac{1}{[(1+2\bar{s}^2)D]^{1/2}} \tag{44}$$

and the spherical condition becomes

$$Z_{\mathrm{SP}}(0)\bar{x}_\alpha=\frac{K}{\sqrt{2}}\qquad \alpha=1,2 \tag{45}$$

while from (42):

$$q_{\mathrm{SG}}=\frac{\sqrt{2}}{K}Z_{\mathrm{SP}}(0)\bar{q}. \tag{46}$$

The subscript SP in a function indicates its value at the saddle point while D in (44) is the determinant of the second derivatives of G:

$$D=(1-2\bar{x}_1\bar{x}_2-2\bar{q}^2)[(1-2\bar{x}_1^2)(1-2\bar{x}_2^2)-4\bar{q}^4]-8\bar{q}^2[\bar{x}_1^2+\bar{x}_2^2+4\bar{x}_1\bar{x}_2(\bar{q}^2-\bar{x}_1\bar{x}_2)]. \tag{47}$$

The bar over the variables indicates the solutions of the saddle-point equations obtained from (40) or from (37) when $\boldsymbol{\theta}=0$:

$$\bar{x}_1=(z-\bar{x}_2)/2|\mathbf{M}| \tag{48a}$$

$$\bar{x}_2=(z-\bar{x}_1)/2|\mathbf{M}| \tag{48b}$$

$$\bar{q}=\bar{q}/2|\mathbf{M}| \tag{49}$$

$$\mathrm{i}\bar{s}=1/2(z-\mathrm{i}\bar{s}) \tag{50}$$

$$|\mathbf{M}|=(z-\bar{x}_1)(z-\bar{x}_2)-\bar{q}^2. \tag{51}$$

For $T>T_{\mathrm{c}}$ and $K=\beta J$ small, z and $|\mathbf{M}|$ take large values from (45) and (48). Then from (49)

$$\bar{q}=0 \tag{52}$$

and

$$\bar{x}_1=\bar{x}_2=\mathrm{i}\bar{s}=\frac{z}{2}-\left(\frac{z^2}{4}-\frac{1}{2}\right)^{1/2}. \tag{53}$$

By introducing the results of (52) and (53) in (40), (44) and (47) one verifies that

$$Z_{\mathrm{SP}}(0)=1 \tag{54}$$

thus proving the consistency of the theory. Equation (53), together with the spherical condition in (45), allow us to solve for z as a function of K:

$$z=\frac{1}{\sqrt{2}}\left(K+\frac{1}{K}\right)\qquad K<1. \tag{55}$$

At the critical temperature $K_c=1$ we also have $z_c=\sqrt{2}$ and $2|\mathbf{M}|_c=1$. For $K>K_c$, z 'sticks' to the value z_c and we find that the only real solutions are still given by (52) and (53) with $z=z_c$, while the spherical condition in (45) no longer holds.

The free energy per site is given by

$$\beta F=\beta\Omega-(K/\sqrt{2})z \tag{56}$$

where from (14), (15) and (45) $\beta\Omega$ is the solution of

$$\frac{\partial}{\partial z}\beta\Omega(z)=\bar{x}_1=\frac{z}{2}-\left(\frac{z^2}{4}-\frac{1}{2}\right)^{1/2} \tag{57}$$

with the boundary condition

$$\lim_{K\to 0}[\beta\Omega(z)]=\lim_{K\to 0}\tfrac{1}{2}\ln\left(\frac{K}{\sqrt{2}}z\right). \tag{58}$$

We obtain for $\beta\Omega(z)$ by integrating (57)

$$\beta\Omega(z)=\frac{z}{2}\left[\frac{z}{2}-\left(\frac{z^2}{4}-\frac{1}{2}\right)^{1/2}\right]-\tfrac{1}{2}\ln\left[\frac{z}{2}-\left(\frac{z^2}{4}-\frac{1}{2}\right)^{1/2}\right]+\tfrac{1}{2}\ln\left(\frac{K}{2\sqrt{2}}\right)-\tfrac{1}{4} \tag{59}$$

which gives for the free energy in (56)

$$\beta F=\begin{cases}-\tfrac{1}{4}K^2-\tfrac{1}{2}(1+\ln(2)) & T>T_c\\ -K+\tfrac{1}{2}\ln(K/2)+\tfrac{1}{4} & T<T_c\end{cases} \tag{60}$$

and a negative divergent entropy at low temperature:

$$S(T)\approx -k_B\ln\left(\frac{J}{2k_BT}\right). \tag{61}$$

From (18) and (39) the uniform susceptibility is

$$\chi=\frac{\sqrt{2}}{J}(\bar{x}_1-\bar{q})=\begin{cases}\beta & \text{if } T>T_c\\ 1/J & \text{if } T<T_c\end{cases}. \tag{62}$$

Equations (60)-(62) are the results of Kosterlitz *et al* (1976). We believe there is a printing error in (7) of this reference for $T<T_c$ as the quoted results for the low temperature entropy and specific heat are obtained from $\ln(J/2T)$ in place of $\ln(T/2J)$.

The present exact solution using supermathematics explains some pathological aspects of the model. In spite of the complexity of the method the result of (39) is deceptively simple. $Z(\{h\})$ reduces to the partition functional for only two 'replicas' that are effectively uncoupled because the solution to the saddle-point equations is $\bar{q}=0$ for all temperatures. The parameter s associated to $\langle\chi^*\chi\rangle$ only plays the role of normalising $Z(0)$ to unity. In fact, it is trivial to show that the 'very wrong' solution that considers only one replica:

$$\beta\bar{F}=-\ln\langle Q(\mu)\rangle_{CA}-\mu \tag{63}$$

also reproduces the exact result. Following the same steps that lead to (27) we obtain

$$\langle Q(\mu)\rangle_{CA}=\int_{-\infty}^{\infty}\frac{dx}{(2\pi/N)^{1/2}}e^{-N\bar{G}} \tag{64}$$

where

$$\bar{G}=\tfrac{1}{2}[x^2+\ln(z-x)+\ln(K/\sqrt{2})]. \tag{65}$$

The saddle-point equation, together with the spherical condition in (4), gives

$$\bar{x}=\frac{1}{2(z-\bar{x})}=\frac{z}{2}-\left(\frac{z^2}{4}-\frac{1}{2}\right)^{1/2}=\begin{cases}K/\sqrt{2} & \text{if } K<1\\ 1/\sqrt{2} & \text{if } K\geqslant 1\end{cases}. \tag{66}$$

It follows that

$$\beta\bar{F}=\bar{G}_{\text{SP}}-(K/\sqrt{2})z \tag{67}$$

coincides with the exact result of (60). The same calculation with a uniform magnetic field gives the result of (62) for the susceptibility.

To end this section we comment on the introduction of the Grassmann fields θ_α, θ_α^* in the derivation of (27). The fact that these fields are so easily integrated using (38) shows that they are unnecessary. Indeed, we can use (23) to write in (20)

$$\exp\left(\frac{K^2}{2N}\sum_\alpha \eta_\alpha^*\eta_\alpha\right)=1+\frac{K^2}{2N}\sum_\alpha \eta_\alpha^*\eta_\alpha+\left(\frac{K^2}{2N}\right)^2\eta_1^*\eta_1\eta_2^*\eta_2 \tag{68}$$

and explicit integration of the $\sigma_{i\alpha}$, χ_i^*, χ_i will reproduce (39). The reason we used (26) instead was to have more insight into the usual procedure (Efetov 1983, Verbaarschot and Zirnbauer 1985) of introducing a superfield order parameter as in (34) that satisfies the saddle-point equation (37). The use of (68) would not simplify the calculations, but it shows that at least in the present problem the superfield order parameter is only a mathematical artifice. Equation (38) is exact and it tells us that the only solution to the saddle point (37) is the trivial one $\theta_\alpha=\theta_\alpha^*=0$.

Finally, we want to point out that, although the method of supermathematics may look too involved in the present problem as compared with the elegant solution of Kosterlitz *et al* (1976), what we are doing in fact is giving an alternative derivation of Wigner's semicircular law.

Indeed, the eigenvalue density of the random matrix **K** in (3) is written from (1):

$$\begin{aligned}\rho(\omega)&=-\frac{1}{\pi}\,\text{Im}\left(\frac{\partial}{\partial\mu}\frac{1}{N}\langle\ln Q(\mu)\rangle_{\text{CA}}\right)_{\mu=\omega/2-\mathrm{i}\eta}\\ &=\frac{1}{\pi}\,\text{Im}\left(\frac{\partial}{\partial\mu}\beta\Omega(\mu)\right)_{\mu=\omega/2-\mathrm{i}\eta}\end{aligned} \tag{69}$$

which gives from (57) and $z=(\sqrt{2}/K)\mu$

$$\begin{aligned}\rho(\omega)&=\frac{1}{\pi K^2}\,\text{Im}[\tfrac{1}{2}\omega-(\tfrac{1}{4}\omega^2-K^2-\mathrm{i}\eta)^{1/2}]\\ &=\frac{1}{2\pi K^2}(4K^2-\omega^2)^{1/2}\theta(4K^2-\omega^2).\end{aligned} \tag{70}$$

3. Discussion

In the present paper we have given one more application of supermathematics to the theory of condensed matter by solving the mean spherical model of a spin glass. We recovered the exact results of Kosterlitz *et al* (1976) and moreover we showed that the exact solution corresponds to uncoupled replicas, thus exhibiting the pathology of the model. One may wonder why we obtain a spin glass transition with order parameter $q_{\text{SG}}=0$ above *and* below the critical temperature. This is because q_{SG} was defined in (17) as $\langle\langle\sigma_i\rangle^2{}_{\rm TA}\rangle_{\rm CA}$ and $\langle\sigma_i\rangle_{\rm TA}$ equals zero in the spherical model, as pointed out by Berlin and Kac (1952). Then our $q_{\rm SG}$ is not the correct order parameter of the theory; the appropriate order parameter should involve rather $\langle|\sigma_i|\rangle_{\rm TA}$.

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References

Bell J A 1962 *Lectures on the Many-Body Problem* ed E R Caianello (New York: Academic) pp 81-9
Berezin F A 1966 *The Method of Second Quantization* (New York: Academic)
Berlin T H and Kac M 1952 *Phys. Rev.* **86** 821-35
Brezin E, Gross D J and Itzykson C 1984 *Nucl. Phys.* B **235** [FS11] 24-44
De Dominicis C 1978 *Phys. Rev.* B **18** 4913-9
de Almeida J R L, Jones R C, Kosterlitz J M and Thouless D J 1978 *J. Phys. C: Solid State Phys.* **11** L871-5
Edwards S F and Jones R C 1976 *J. Phys. A: Math. Gen.* **9** 1595-603
Edwards S F and Sherrington D 1967 *Proc. Phys. Soc.* **90** 3-22
Efetov K B 1983 *Adv. Phys.* **32** 53-127
Joyce G S 1972 *Phase Transitions and Critical Phenomena* vol 2 ed C Domb and M S Green (New York: Academic) pp 375-442
Jug G 1984 *Phys. Rev. Lett.* **53** 9-12
Klein A and Perez J F 1984 *Nucl. Phys.* B to be published
Kosterlitz J M, Thouless D J and Jones R C 1976 *Phys. Rev. Lett.* **36** 1217-20
Mehta M L 1967 *Random Matrices and the Statistical Theory of Energy Levels* (New York: Academic) p 240
Parisi G and Sourlas N 1979 *Phys. Rev. Lett.* **43** 744-7
Pastur L A 1982 *J. Stat. Phys.* **27** 119-51
Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **36** 1217-20
Verbaarschot J J M and Zirnbauer M R 1984 *Ann. Phys., NY* **158** 78-119
—— 1985 *J. Phys. A: Math. Gen.* **17** 1093-109
Wegner F 1983 *Z. Phys.* B **51** 279-85